2016-12-01

Some Spectral Properties of a Quantum Field Theoretic Hamiltonian

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

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We derive the ground-state eigenvalues and eigenvectors for a simplified version of the 1-D QED single electron-photon model that Glasgow et al recently developed [2]. This model still allows for meaningful interaction between electrons and photons while keeping only the minimum needed to do so. We investigate the interesting spectral properties of this model. We determine that the eigenvectors are orthogonal as one would expect and normalize them.

Keywords: spectral properties, quantum field theory, electron, photon

Contents

1 Introduction

Quantum Field Theory is one of the more important characterizations of Quantum Mechanics created in the last 100 years in physics (although we reduce QFT to basic Quantum Mechanics). It is the representation of physical objects by fields rather than as a wave or a particle. This is useful because it gets around the concepts that are hard to grasp, such as wave-particle duality. An important subset of Quantum Field Theory is known as Quantum Electrodynamics. Quantum Field Theory is important because it is necessary for the reconciliation of Quantum Mechanics with Special Relativity [4].

We will be using a few of the concepts from Quantum Field Theory. The concept of Creation and Annihilation operators, a simplified version of the QED Hamiltonian as described by Cohen-Tannoudji [1], and bra-ket notation.

2 Bra-ket Notation and Creation/Annihilation

We will first define the notation as used for bras and kets. A bra looks like $\langle p|$ and a ket looks like $|p\rangle$. An informal way of thinking about bras and kets is to think of a ket as a vector and a bra as its transpose, so that a bra applied on a ket is like an inner product. A more formal definition of them is to treat a ket as an element of a Hilbert space and treat a bra as an element of its dual-space. In other words we can think of a bra as a linear functional that acts on a ket. So when we apply a bra-ket pair we have $\langle p|q\rangle$ and we should get a number out since the ket $\langle p|$ is a linear functional. Next we look at how bras and kets work together. This is done using the concept of creation and annihilation for our model. We think of a bra as annihilation of an electron with momentum p and a ket as creation of an electron with momentum p (if it is of the form $|p\rangle$). Now we will look at exactly how they interact

$$\langle p|q\rangle = \delta(p-q),$$

where $\delta(p-q)$ is the Dirac delta function.

Also we now define the blank ket $|\rangle$ to be a "bare" electron or in other words an electron without a photon. We say that this blank ket has the following inner product with itself

$$\langle|\rangle = 1.$$

Also we note that this bare electron has the following property as well when applied to a ket of the following form

$$\langle p|\rangle = 0.$$

So in other words the blank ket and a dressed electron are orthogonal to each other. These properties are very useful in our calculations and follow immediately from the relationship between inner products and measurement in Quantum Mechanics. The calculations we will do include finding the form of the eigenvectors (and associated eigenvalues) for our Hamiltonian. It will also include determining if the eigenvectors that we find are orthogonal to each other as we would expect. This is where the orthogonality of the blank ket and the dressed electron will be useful. Also in the future we will be able to show completeness of these eigenvectors. Completeness will be useful so that we can spectrally decompose our Hamiltonian allowing for dynamics using our model.

3 Hamiltonian

We will define the Hamiltonian as

$$H = E_0|\rangle\langle| + \int d\alpha E(\alpha)|\alpha\rangle\langle\alpha| + \int d\alpha g(\alpha)(|\alpha\rangle\langle| + |\rangle\langle\alpha|), \tag{1}$$

where E_0 is the energy associated with a bare electron with momentum 0, $E(\alpha)$ is the energy associated with an dressed electron with momentum α, and $g(\alpha)$ is the coupling between electrons and photons.

Note that

$$H|\alpha\rangle = \int d\alpha' E(\alpha')|\alpha'\rangle\delta(\alpha - \alpha') + \int d\alpha' g(\alpha')|\rangle\delta(\alpha - \alpha')$$
$$= E(\alpha)|\alpha\rangle + g(\alpha)|\rangle$$

and

$$H|\rangle = E_0|\rangle\langle|\rangle + \int d\alpha' g(\alpha')|\alpha\rangle\langle|\rangle$$
$$= E_0|\rangle + \int d\alpha g(\alpha)|\alpha\rangle.$$

Notice that both $|\rangle$ and $|\alpha\rangle$ change a fair amount when acted on by the Hamiltonian H. $|\rangle$ becomes itself plus the integral of the coupling integrated over $|\alpha\rangle$, and $|\alpha\rangle$ becomes itself plus the coupling multiplied by $|\rangle$. This means that H perturbs a state of one type into another. Therefore in order to find eigenvectors we imagine they will start as some sort of linear combination of $|\rangle$ and $|\alpha\rangle$ already.

4 Eigenvectors

Recall equation (1) and note the following

$$E(-\alpha) = E(\alpha)$$
$$g(-\alpha) = -g(\alpha)$$
$$g(0) = 0$$
$$E_0 < E_{min} := min_{\alpha \in R} E(\alpha) = E(0).$$

where $E(\alpha)$ is unbounded from above and monotonically increasing.

Note that when $g = 0$ we get

$$H|\alpha\rangle = E(\alpha)|\alpha\rangle$$

$$H|\rangle = E_0|\rangle.$$

By the symmetries of $E(\alpha)$ and $g(\alpha)$ we get

$$H|\alpha\rangle_{g,+} := H(|\alpha\rangle + |-\alpha\rangle)$$

$$= E(\alpha)|\alpha\rangle + g(\alpha)|\rangle + E(-\alpha)|-\alpha\rangle + g(-\alpha)|\rangle$$

$$= E(\alpha)|\alpha\rangle + g(\alpha)|\rangle + E(\alpha)|-\alpha\rangle + -g(\alpha)|\rangle$$

$$= E(\alpha)(|\alpha\rangle + |-\alpha\rangle), \alpha \geq 0$$

and we have half of the "upper" eigenvectors are of this form. We try a simple form for the other half and find that

$$H(|\alpha\rangle - |-\alpha\rangle)$$

$$= E(\alpha)|\alpha\rangle + g(\alpha)|\rangle - E(-\alpha)|-\alpha\rangle - g(-\alpha)|\rangle$$

$$= E(\alpha)|\alpha\rangle + g(\alpha)|\rangle - E(\alpha)|-\alpha\rangle + g(\alpha)|\rangle$$

$$= E(\alpha)(|\alpha\rangle - |-\alpha\rangle) + 2g(\alpha)|\rangle.$$

which are not eigenvectors so instead we try an eigenvector of the form

$$|\alpha\rangle_{g,-} = |\alpha\rangle - |-\alpha\rangle + c(\alpha)|\rangle + \int d\beta f_\alpha(\beta)|\beta\rangle, \alpha > 0.$$

If we try this form we get issues with $f_\alpha(\beta)$ being singular near $+\alpha, -\alpha$ because of the symmetries noted above. A way to fix this is to have an eigenvector of the form

$$|\alpha\rangle_{g,-} = |\alpha\rangle - |-\alpha\rangle + c(\alpha)|\rangle + \int_{\beta>0} d\beta f_\alpha(\beta)(|\beta\rangle - |\alpha\rangle) + \int_{\beta>0} d\beta f_\alpha(\beta)(|\beta\rangle - |-\alpha\rangle)$$

$$= |\alpha\rangle - |-\alpha\rangle + c(\alpha)|\rangle + \int_{\beta>0} d\beta f_\alpha(\beta)(|\beta\rangle - \theta_+(\beta)|\alpha\rangle - \theta_-(\beta)|-\alpha\rangle), \alpha > 0$$

where

$$\theta_\pm(\beta) := \theta(\pm\beta) := \begin{cases} 1 & \text{if } \pm\beta > 0 \\ 0 & \text{if } \pm\beta < 0 \end{cases}$$

so that when $f_\alpha(\beta)$ is singular, it is multiplied by the zero vector.

Now applying the Hamiltonian on $|\alpha\rangle_{g,-}$ we get

$$H|\alpha\rangle_{g,-} = H(|\alpha\rangle - |-\alpha\rangle) + c(\alpha)H|\rangle + \int d\beta f_\alpha(H|\beta\rangle - \theta_+(\beta)H|\alpha\rangle - \theta_-(\beta)H|-\alpha\rangle)$$

$$= E(\alpha)(|\alpha\rangle - |-\alpha\rangle) + 2g(\alpha)|\rangle + c(\alpha)(E_0|\rangle + \int d\beta g(\beta)|\beta\rangle)$$

$$+ \int d\beta f_\alpha(\beta)(E(\beta)|\beta\rangle + g(\beta)|\rangle - \theta_+(\beta)(E(\alpha)|\alpha\rangle + g(\alpha)|\rangle) - \theta_-(\beta)(E(\alpha)|-\alpha\rangle - g(\alpha)|\rangle))$$

$$= E(\alpha)(|\alpha\rangle - |-\alpha\rangle) + (2g(\alpha)|\rangle - E_0 c(\alpha) + \int d\beta f_\alpha(\beta)(g(\beta) - (\theta_+(\beta) - \theta_-(\beta))g(\alpha)))|\rangle$$

$$+ \int d\beta((f_\alpha(\beta)E(\beta) + g(\beta)c(\alpha)|\beta\rangle - E(\alpha)f_\alpha(\beta)(\theta_+(\beta)|\alpha\rangle + \theta_-(\beta)|-\alpha\rangle))$$

$$= E_{g,-}(\alpha)|\alpha\rangle_{g,-}$$

$$= E_{g,-}(\alpha)(|\alpha\rangle - |-\alpha\rangle) + c(\alpha)E_{g,-}(\alpha)|\rangle + \int d\beta f_\alpha(\beta)(E_{g,-}(\alpha)|\beta\rangle$$

$$-E_{g,-}(\alpha)(\theta_+(\beta)|\alpha\rangle + \theta_-(\beta)|-\alpha\rangle))$$

which holds iff

$$E_{g,-}(\alpha) = E(\alpha)$$

$$E(\alpha)c(\alpha) = 2g(\alpha) - E_0c(\alpha) + \int d\beta f_\alpha(\beta)(g(\beta) - (\theta_+(\beta) - \theta_-(\beta))g(\alpha))$$

$$= 2g(\alpha) - E_0c(\alpha) + \int d\beta f_\alpha(\beta)(g(\beta) - sign(\beta)g(\alpha))$$

$$f_\alpha(\beta)E(\beta) + g(\beta)c(\alpha) = f_\alpha(\beta)E(\alpha)$$

or

$$c(\alpha) = 2\,\frac{g(\alpha)}{E(\alpha) - E_0 + \int d\beta \frac{sign(\beta)g(\beta)}{E(\beta)-E(\alpha)}(sign(\beta)g(\beta) - sign(\beta)sign(\beta)g(\alpha))}$$

$$= 2\,\frac{g(\alpha)}{E(\alpha) - E_0 + \int d\beta \frac{g(|\beta|)}{E(\beta)-E(\alpha)}(g(|\beta|) - g(\alpha))}$$

$$= 2\,\frac{g(\alpha)}{E(\alpha) - E_0 + 2\int_{\beta>0} d\beta g(\beta)\frac{g(\beta)-g(\alpha)}{E(\beta)-E(\alpha)}}$$

and finally we get

$$|\alpha\rangle_{g,-} = |\alpha\rangle - |-\alpha\rangle + c(\alpha)(|\rangle + \int d\beta f_\alpha(\beta)(|\beta\rangle - \theta_+(\beta)|\alpha\rangle - \theta_-(\beta)|-\alpha\rangle))$$

$$= |\alpha\rangle - |-\alpha\rangle + c(\alpha)(|\rangle - \int d\beta \frac{g(\beta)}{E(\beta) - E(\alpha)}(|\beta\rangle - \theta_+(\beta)|\alpha\rangle - \theta_-(\beta)|-\alpha\rangle))$$

$$= |\alpha\rangle - |-\alpha\rangle + c(\alpha)(|\rangle - \int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta) - E(\alpha)}(|\beta\rangle - |\alpha\rangle)$$

$$- \int_{\beta<0} d\beta \frac{g(\beta)}{E(\beta) - E(\alpha)}(|\beta\rangle - |-\alpha\rangle))$$

$$= |\alpha\rangle - |-\alpha\rangle + c(\alpha)(|\rangle - \int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta) - E(\alpha)}(|\beta\rangle - |\alpha\rangle))$$

$$-\int_{\beta>0} d\beta \frac{g(-\beta)}{E(-\beta)-E(\alpha)}(|-\beta\rangle - |-\alpha\rangle))$$

$$= |\alpha\rangle - |-\alpha\rangle + c(\alpha)(|\rangle - \int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}(|\beta\rangle - |-\beta\rangle - |\alpha\rangle + |-\alpha\rangle))$$

where

$$c(\alpha) = 2\frac{g(\alpha)}{E(\alpha) - E_0 + 2\int_{\beta>0} d\beta g(\beta)\frac{g(\beta)-g(\alpha)}{E(\beta)-E(\alpha)}}.$$

This gives us a well-defined state $|\alpha\rangle_{g,-}$ that satisfies the eigenvalue-eigenvector equation, with eigenvalue the same as for an unperturbed state $|\alpha\rangle$. Note that if we have a state

$$|\Psi\rangle = \int d\alpha \Psi(\alpha)|\alpha\rangle.$$

Then we can look at the inner product of it with $|\alpha\rangle_{g,-}$

$$(|\Psi\rangle, |\alpha\rangle_{g,-}) = \int d\alpha \Psi^*(\alpha')\langle\alpha'|(|\alpha\rangle - |-\alpha\rangle + c(\alpha)(|\rangle$$

$$-\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}(|\beta\rangle - |-\beta\rangle - |\alpha\rangle + |-\alpha\rangle)$$

$$= \int d\alpha \Psi^*(\alpha')(\delta(\alpha'-\alpha) - \delta(\alpha'+\alpha) - c(\alpha)$$

$$\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}(\delta(\alpha'-\beta) - \delta(\alpha'+\beta) - (\delta(\alpha'-\alpha) - \delta(\alpha'+\alpha)))$$

$$= \Psi^*(\alpha) - \Psi^*(-\alpha) - c(\alpha)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}(\Psi^*(\beta) - \Psi^*(-\beta) - \Psi^*(\alpha) + \Psi^*(-\alpha))$$

$$= \Psi^*(\alpha) - \Psi^*(-\alpha) - c(\alpha)\int_{\beta>0} d\beta g(\beta)\frac{\Psi^*(\beta)-\Psi^*(\alpha)}{E(\beta)-E(\alpha)} + c(\alpha)\int_{\beta>0} d\beta g(\beta)\frac{\Psi^*(-\beta)-\Psi^*(-\alpha)}{E(-\beta)-E(-\alpha)}.$$

This is well-defined for a fairly large class of functions $\Psi(\alpha)$ which means that $|\alpha\rangle_{g,-}$ is a well-defined linear functional for that class of functions.

Another important aside to mention is the fact that the eigenvalues for these $|\alpha\rangle_{g,+}$ and $|\alpha\rangle_{g,-}$ are both $E(\alpha)$. This is interesting because it is the energy associated with an unperturbed state. Standard Perturbation Theory typically gives us an increase in the value for the eigenvalues of the perturbed state for the "upper" eigenvalues. Our problem sees no change in the energy values. For additional discussion on this see Van Hove [3].

Recall again equation (1). We now try to find an eigenvector corresponding to a perturbed $|\rangle$. We assume it to be of the form

$$|\rangle_g = |\rangle + d(\beta)|\beta\rangle + \int d\alpha' G(\alpha';\beta)|\alpha'\rangle.$$

We then apply H onto $|\rangle_g$

$$H|\rangle_g = H|\rangle + d(\beta)H|\beta\rangle + \int d\alpha' G(\alpha';\beta)H|\alpha'\rangle$$

$$= E_0|\rangle + \int d\alpha g(\alpha)|\alpha\rangle + \int d\alpha E(\alpha)|\alpha\rangle\delta(\beta-\alpha)d(\beta) + \int d\alpha g(\alpha)d(\beta)|\rangle\delta(\beta-\alpha)$$

$$+ \int d\alpha \int d\alpha' E(\alpha)G(\alpha';\beta)|\alpha\rangle\delta(\alpha'-\alpha) + \int d\alpha \int d\alpha' g(\alpha)G(\alpha';\beta)|\rangle\delta(\alpha'-\alpha)$$

$$= E_0|\rangle + \int d\alpha g(\alpha)|\alpha\rangle + E(\beta)d(\beta)|\beta\rangle + g(\beta)d(\beta)|\rangle + \int E(\alpha)G(\alpha;\beta)|\alpha\rangle + \int d\alpha g(\alpha)G(\alpha;\beta)|\rangle.$$

Applying λ onto $|\rangle_g$ we get

$$\lambda|\rangle_g = \lambda E_0|\rangle + \lambda d(\beta)|\beta\rangle + \int d\alpha' \lambda G(\alpha';\beta)|\alpha'\rangle.$$

Which leads to the following system of equations for λ, $d(\beta)$, $G(\alpha;\beta)$

$$\lambda E_0 = E_0 + g(\beta)d(\beta) + \int d\alpha g(\alpha)G(\alpha;\beta)$$

$$\lambda d(\beta) = E(\beta)d(\beta)$$

$$\lambda G(\alpha;\beta) = g(\alpha) + E(\alpha)G(\alpha;\beta).$$

and we choose $d(\beta) = 0$ because if we do not we end up with the same eigenvectors as before.

$$\lambda = E_0 + \int d\alpha g(\alpha)G(\alpha)$$

$$\lambda G(\alpha) = g(\alpha) + E(\alpha)G(\alpha)$$

$$G(\alpha) = \frac{g(\alpha)}{\lambda - E(\alpha)}$$

$$\lambda = E_0 - \int d\alpha \frac{g^2(\alpha)}{E(\alpha) - \lambda}.$$

Thus we find that $|\rangle_g$ is

$$|\rangle_g = |\rangle + \int d\alpha \frac{g(\alpha)}{\lambda - E(\alpha)}|\alpha\rangle.$$

An important side note is whether λ is well-defined in the equation

$$\lambda = E_0 - \int d\alpha \frac{g^2(\alpha)}{E(\alpha) - \lambda}.$$

Note that left-hand side will be a straight line on a graph (if plotted versus itself), while on the right-hand side if we start λ down at $-\infty$ and move up we start at close to E_0 on the right-hand side and push down closer and closer to the E_0 we are looking for. However when we start from $-\infty$ and increase λ, then we have that it is increasing on the left-hand side, so they must cross somewhere since both sides consist of smooth functions. Therefore there must be at least one root to this equation. We pick the smallest root to be our value of λ and note that this smallest root is the classical one (i.e. one where we do not need to use principal values).

5 Orthonormality of Eigenvectors

Now we would like to check if the eigenvectors we have found are an orthonormal set. We first look at a nicer form of $|\alpha\rangle_{g,-}$ by multiplying by the denominator of $c(\alpha)$ onto it.

$$(E(\alpha) - E_0 + 2\int_{\beta>0} d\beta g(\beta)\frac{g(\beta)-g(\alpha)}{E(\beta)-E(\alpha)})|\alpha\rangle_{g,-}$$

$$= E(\alpha)|\alpha\rangle - E(\alpha)|-\alpha\rangle - E_0|\alpha\rangle + E_0|-\alpha\rangle + 2\int_{\beta>0} d\beta g(\beta)\frac{g(\beta)-g(\alpha)}{E(\beta)-E(\alpha)}(|\alpha\rangle + |-\alpha\rangle)$$

$$+2g(\alpha)(\,|\rangle - \int_{\beta>0} d\beta\frac{g(\beta)}{E(\beta)-E(\alpha)}(|\beta\rangle - |-\beta\rangle - |\alpha\rangle + |-\alpha\rangle))$$

$$= (E(\alpha)-E_0)(|\alpha\rangle - |-\alpha\rangle) + 2g(\alpha)|\rangle + 2\int_{\beta>0} d\beta\frac{g^2(\beta)}{E(\beta)-E(\alpha)}(|\alpha\rangle - |-\alpha\rangle)$$

$$-2\int_{\beta>0} d\beta\frac{g(\alpha)g(\beta)}{E(\beta)-E(\alpha)}(|\beta\rangle - |-\beta\rangle)$$

$$= (E(\alpha)-E_0)(|\alpha\rangle - |-\alpha\rangle) + 2g(\alpha)|\rangle$$

$$+2\int_{\beta>0} d\beta g(\beta)\frac{g(\beta)(|\alpha\rangle - |-\alpha\rangle) - g(\alpha)(|\beta\rangle - |-\beta\rangle)}{E(\beta)-E(\alpha)}.$$

This form will be much easier to work with than the standard $|\alpha\rangle_{g,-}$ and we will call it $k|\alpha\rangle_{g,-}$ for simplicity.

Now we check if $|\alpha\rangle_{g,-}$ and $|\rangle_g$ are orthogonal.

$$(E(\alpha) - E_0 + 2\int_{\beta>0} d\beta g(\beta)\frac{g(\beta)-g(\alpha)}{E(\beta)-E(\alpha)})\ {}_{g,-}\langle\alpha|\rangle_g = 2g(\alpha)$$

$$-(E(\alpha)-E_0)\int d\alpha' \frac{g(\alpha')}{E(\alpha')-E_g}\delta(\alpha-\alpha') + (E(\alpha)-E_0)\int d\alpha' \frac{g(\alpha')}{E(\alpha')-E_g}\delta(\alpha+\alpha')$$

$$+2\int_{\beta>0} d\beta \int d\alpha' \frac{g(\beta)g(\alpha)}{E(\beta)-E(\alpha)}\frac{g(\alpha')}{E(\alpha')-E_g}\delta(\alpha'-\beta)$$

$$-2\int_{\beta>0} d\beta \int d\alpha' \frac{g(\beta)g(\alpha)}{E(\beta)-E(\alpha)}\frac{g(\alpha')}{E(\alpha')-E_g}\delta(\alpha'+\beta)$$

$$-2\int_{\beta>0} d\beta \int d\alpha' \frac{g^2(\beta)}{E(\beta)-E(\alpha)}\frac{g(\alpha')}{E(\alpha')-E_g}\delta(\alpha'-\alpha)$$

$$+2\int_{\beta>0} d\beta \int d\alpha' \frac{g^(\beta)}{E(\beta)-E(\alpha)}\frac{g(\alpha')}{E(\alpha')-E_g}\delta(\alpha'+\alpha)$$

$$= 2g(\alpha) - 2(E(\alpha)-E_0)\frac{g(\alpha)}{E(\alpha)-E_g} - 4\int_{\beta>0} d\beta \frac{g^2(\beta)}{E(\beta)-E(\alpha)}\frac{g(\alpha)}{E(\alpha)-E_g}$$

$$+4\int_{\beta>0} d\beta \frac{g^2(\beta)}{E(\beta)-E(\alpha)}\frac{g(\alpha)}{E(\beta)-E_g}$$

$$= 2g(\alpha) - 2(E(\alpha)-E_0)\frac{g(\alpha)}{E(\alpha)-E_g} + 4\int_{\beta>0} d\beta \frac{g^2(\beta)g(\alpha)}{E(\beta)-E(\alpha)}\frac{(E(\beta)-E_g)-(E(\alpha)-E_g)}{(E(\beta)-E_g)(E(\alpha)-E_g)}$$

$$= 2g(\alpha) - 2(E(\alpha)-E_0)\frac{g(\alpha)}{E(\alpha)-E_g} + 4\int_{\beta>0} d\beta \frac{g^2(\beta)g(\alpha)}{E(\beta)-E(\alpha)}\frac{E(\beta)-E(\alpha)}{(E(\beta)-E_g)(E(\alpha)-E_g)}$$

$$= 2g(\alpha) - 2(E(\alpha)-E_0)\frac{g(\alpha)}{E(\alpha)-E_g} + 4\int_{\beta>0} d\beta \frac{g^2(\beta)g(\alpha)}{(E(\beta)-E_g)(E(\alpha)-E_g)}$$

$$= 2g(\alpha) - \frac{g(\alpha)}{E(\alpha)-E_g}(2(E(\alpha)-E_0) - 4\int_{\beta>0} d\beta \frac{g^2(\beta)}{E(\beta)-E_g})$$

$$= 2g(\alpha) - \frac{g(\alpha)}{E(\alpha)-E_g}(2(E(\alpha)-E_0) + 2(E_0-E_g))$$

$$= 2g(\alpha) - \frac{g(\alpha)}{E(\alpha)-E_g}2(E(\alpha)-E_g) = 2g(\alpha) - 2g(\alpha) = 0.$$

Thus we have that $|\alpha\rangle_{g,-}$ and $|\rangle_g$ are orthogonal as desired.

Note that

$$_g\langle|\rangle_g = \langle|\rangle + \int d\alpha \int d\alpha' \frac{g(\alpha)}{(E(\alpha)-E_g)} \frac{g(\alpha')}{(E(\alpha')-E_g)} \langle\alpha|\alpha'\rangle$$

$$= {E_0}^2 + \int d\alpha \int d\alpha' \frac{g(\alpha)}{(E(\alpha)-E_g)} \frac{g(\alpha')}{(E(\alpha')-E_g)} \delta(\alpha-\alpha')$$

$$= {E_0}^2 + \int d\alpha \frac{g^2(\alpha)}{(E(\alpha)-E_g)^2}.$$

Thus if we divide $|\rangle_g$ by

$$\sqrt{1+\int d\alpha \frac{g^2(\alpha)}{(E(\alpha)-E_g)^2}}$$

we get an orthonormal vector.

Now we check if $|\alpha\rangle_{g,-}$ is orthogonal to another vector of its same type i.e. $|\gamma\rangle_{g,-}$

$$_{g,-}\langle\gamma|\alpha\rangle_{g,-} = [\langle\gamma| - \langle-\gamma| + c(\gamma)(\ \langle| - \int_{\rho>0} d\rho \frac{g(\rho)}{E(\rho)-E(\gamma)} (\langle\rho| - \langle-\rho| - \langle\gamma| + \langle-\gamma|))][|\alpha\rangle$$

$$-|-\alpha\rangle + c(\alpha)(\ |\rangle - \int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)} (|\beta\rangle - |-\beta\rangle - |\alpha\rangle + |-\alpha\rangle))]$$

$$= 2\delta(\gamma-\alpha) - 2\delta(\gamma+\alpha) + c(\alpha)c(\gamma)$$

$$-c(\gamma)\int_{\rho>0} d\rho \frac{g(\rho)}{E(\rho)-E(\gamma)} (2\delta(\rho-\alpha) - 2\delta(\rho+\alpha) - 2\delta(\gamma-\alpha) + 2\delta(\gamma+\alpha))$$

$$-c(\alpha)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)} (2\delta(\beta-\gamma) - 2\delta(\beta+\gamma) - 2\delta(\alpha-\gamma) + 2\delta(\gamma+\alpha))$$

$$+c(\alpha)c(\gamma)\int_{\beta>0} d\beta \int_{\rho>0} d\rho \frac{g(\beta)}{E(\beta)-E(\alpha)} \frac{g(\rho)}{E(\rho)-E(\gamma)} (2\delta(\rho-\beta) - 2\delta(\rho+\beta)$$

$$-2\delta(\beta-\gamma) + 2\delta(\beta+\gamma) - 2\delta(\rho-\alpha) + 2\delta(\rho+\alpha) + 2\delta(\gamma-\alpha) - 2\delta(\gamma+\alpha))$$

$$= 2\delta(\gamma-\alpha) - 2\delta(\gamma+\alpha) + c(\alpha)c(\gamma) - 2c(\gamma)\frac{g(\alpha)}{E(\alpha)-E(\gamma)} + 2c(\gamma)\frac{g(-\alpha)}{E(-\alpha)-E(\gamma)}$$

$$+c(\gamma)\int_{\rho>0} d\rho \frac{g(\rho)}{E(\rho)-E(\gamma)}(2\delta(\gamma-\alpha) - 2\delta(\gamma-\alpha)) - 2c(\alpha)\frac{g(\gamma)}{E(\gamma)-E(\alpha)}$$

$$+2c(\alpha)\frac{g(-\gamma)}{E(-\gamma)-E(\alpha)} + c(\alpha)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}(2\delta(\gamma-\alpha) - 2\delta(\gamma+\alpha))$$

$$+2c(\alpha)c(\gamma)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(\beta)}{E(\beta)-E(\gamma)} - 2c(\alpha)c(\gamma)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(-\beta)}{E(-\beta)-E(\gamma)}$$

$$-2c(\alpha)c(\gamma)\int_{\rho>0} d\rho \frac{g(\gamma)}{E(\gamma)-E(\alpha)}\frac{g(\rho)}{E(\rho)-E(\gamma)} + 2c(\alpha)c(\gamma)\int_{\rho>0} d\rho \frac{g(-\gamma)}{E(-\gamma)-E(\alpha)}\frac{g(\rho)}{E(\rho)-E(\gamma)}$$

$$-2c(\alpha)c(\gamma)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(\alpha)}{E(\alpha)-E(\gamma)} + 2c(\alpha)c(\gamma)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(-\alpha)}{E(-\alpha)-E(\gamma)}$$

$$+c(\alpha)c(\gamma)\int_{\beta>0} d\beta \int_{\rho>0} d\rho \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(\rho)}{E(\rho)-E(\gamma)}(2\delta(\gamma-\alpha) - 2\delta(\gamma+\alpha))$$

$$= [2\delta(\gamma-\alpha) - 2\delta(\gamma+\alpha)]\Bigg[1 + c(\alpha)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)} + c(\gamma)\int_{\rho>0} d\rho \frac{g(\rho)}{E(\rho)-E(\gamma)}$$

$$+c(\alpha)c(\gamma)\int_{\beta>0} d\beta \int_{\rho>0} d\rho \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(\rho)}{E(\rho)-E(\gamma)}\Bigg]$$

$$+c(\alpha)c(\gamma) - 4c(\gamma)\frac{g(\alpha)}{E(\alpha)-E(\gamma)} - 4c(\alpha)\frac{g(\gamma)}{E(\gamma)-E(\alpha)}$$

$$+4c(\alpha)c(\gamma)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(\beta)}{E(\beta)-E(\gamma)} - 4c(\alpha)c(\gamma)\int_{\rho>0} d\rho \frac{g(\gamma)}{E(\gamma)-E(\alpha)}\frac{g(\rho)}{E(\rho)-E(\gamma)}$$

$$-4c(\alpha)c(\gamma)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}\frac{g(\alpha)}{E(\alpha)-E(\gamma)}$$

$$= [2\delta(\gamma-\alpha) - 2\delta(\gamma+\alpha)]\Bigg[1 + c(\alpha)\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)} + c(\gamma)\int_{\rho>0} d\rho \frac{g(\rho)}{E(\rho)-E(\gamma)}$$

$$+c(\alpha)c(\gamma)\int_{\beta>0} d\beta \int_{\rho>0} d\rho \frac{g(\beta)}{E(\beta)-E(\alpha)} \frac{g(\rho)}{E(\rho)-E(\gamma)}\Bigg] + c(\alpha)c(\gamma) + 4\frac{c(\gamma)g(\alpha)-c(\alpha)g(\gamma)}{E(\gamma)-E(\alpha)}$$

$$+4c(\alpha)c(\gamma)\int_{\beta>0} d\beta g(\beta)\frac{g(\beta)(E(\gamma)-E(\alpha))-g(\gamma)(E(\beta)-E(\alpha))+g(\alpha)(E(\beta)-E(\gamma))}{(E(\beta)-E(\alpha))(E(\beta)-E(\gamma))(E(\gamma)-E(\alpha))}$$

$$= 2\delta(\gamma-\alpha)\Bigg[(E(\alpha)-E_0)(E(\gamma)-E_0) + 2\int_{\beta>0} d\beta \frac{g(\beta)}{E(\beta)-E(\alpha)}(E(\gamma)-E_0)g(\beta)$$

$$+2\int_{\rho>0} d\rho \frac{g(\rho)}{E(\rho)-E(\gamma)}(E(\alpha)-E_0)g(\rho) + 4\int_{\beta>0} d\beta \int_{\rho>0} d\rho \frac{g(\beta)}{E(\beta)-E(\alpha)} \frac{g(\rho)}{E(\rho)-E(\gamma)}\Bigg]$$

This is orthogonal when $\gamma \neq \alpha$ as we would expect (assuming $\alpha > 0$ and $\gamma > 0$).

6 Conclusion

We have found that the eigenvectors do properly exist and correlate to eigenvalues that correspond to the unperturbed states. We have also found that the eigenvectors are orthogonal to each other as we would expect. This is very important for a few reasons. The first being that since they are orthogonal that means they are independent which is another check that they are eigenvectors since all eigenvectors corresponding to distinct eigenvalues are independent. Another reason that it is important that these eigenvectors are orthogonal is for what we want to look at in the future. We want to be able to show completeness of these eigenvectors.

The reason completeness of the eigenvectors is important is because when we have completeness that means we can spectrally decompose our Hamiltonian into its eigenvalues and eigenvectors. Decomposing our Hamiltonian allows for functions to act on our Hamiltonian. This is essential because dynamics only works when we are able to apply an exponential function onto our Hamiltonian and dynamics allows us to do very useful calculations using our model.

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